Serial Number 33

THE COLLEGES OF OXFORD UNIVERSITY

Entrance Examination in Mathematics

MATHEMATICS II

16 November 1993. Morning

Time allowed: 3 hours

Answers to each of Sections A, B, C and D must be attached to separate cover sheets and handed in separately. If no questions are attempted in any one section the cover sheet should still be handed in. Each cover sheet should be clearly labelled A, B, C or D. *All questions cary the same mark. There is no restriction on the number of questions any candidate may attemp but candidates are expected to attempt at least* **three** *questions. Only the best* **five** *solutions will contribute to the total mark for the paper.*

The use of caluclators is allowed, but, unless otherwise stated, exact answers should be given.

Turn Over

SECTION A

A1. (a) In a triangle ABC let I be the intersection point of the angle bisectors of B and C. Prove that

$$\angle BIC = 90^\circ + \frac{1}{2}(\angle CAB).$$

(b) Let O be the point where the perpendicular bisectors of AB, BC and CA intersect. Prove that

$$\angle BOA = 2\angle BCA.$$

[You may find it helpful to use isosceles triangles.]

(c) In a triangle ABC let D be the midpoint of BC and E be the midpoint of AB. Let F be the intersection of AD and CE. Show

$$2FE = FC.$$

[You may find it helpful to use vectors taking C as the origin.]

A2. Consider a curve in the plane, given in polar coordinates (r, θ) by

$$r = \frac{1}{1 - \cos\theta} \quad (0 < \theta < 2\pi).$$

(a) Let O be the origin, P a point on the curve, and let l be the line $x = -1$. Show that the length of OP equals the perpendicular distance from the point P to the line l.

(b) Suppose m is a line through O, which intersects the curve in the two points P_1, P_2. Show that

$$\frac{1}{OP_1} + \frac{1}{OP_2} = 2.$$

(c) Show that for points with coordinates (x, y) on the curve,

$$y^2 = 1 + 2x.$$

A3. The points $A(a_1, a_2)$ and $B(b_1, b_2)$ have position vectors **a** and **b** with respect to a fixed origin **O**. Define

$$\mathbf{a} * \mathbf{b} = a_1 b_1 + a_2 b_2.$$

(a) Prove that:

(i) for a real number k

$$k(\mathbf{a} * \mathbf{b}) = (k\mathbf{a}) * \mathbf{b};$$

(ii)

$$\mathbf{a} * \mathbf{b} = \mathbf{b} * \mathbf{a}.$$

The point C has position vector **c**.

(b) Prove that

$$\mathbf{a} * (\mathbf{b} + \mathbf{c}) = \mathbf{a} * \mathbf{b} + \mathbf{a} * \mathbf{c}.$$

(c) By writing $(a_1, a_2) = (|\mathbf{a}| \cos\alpha, |\mathbf{a}| \sin\alpha)$ and $(b_1, b_2) = (|\mathbf{b}| \cos\beta, |\mathbf{b}| \sin\beta)$, or otherwise, show that

$$\mathbf{a} * \mathbf{b} = |\mathbf{a}|.|\mathbf{b}| \cos\theta,$$

where $|\mathbf{v}|$ denotes the length of a vector $\mathbf{v}$, and θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

A4. Consider a transformation from the plane to itself which takes the point (x, y) to the point (X, Y) where

$$(X, Y) = \left(\frac{-x^2 - y^2 + 1}{x^2 + (1-y)^2}, \frac{-2x}{x^2 + (1-y)^2} \right).$$

(To avoid dividing by zero we do not define the transformation at $(0, 1)$.)

(a) Show that the x-axis transforms to the circle with center (0,0) and radius 1. Show also that this circle transforms to the y-axis.

(b) You are given that the line $y = x \tan\phi$ transforms to a circle C_ϕ with centre on the y-axis. Prove that the radius of C_ϕ is

$$\frac{1}{2} \left| \frac{\cos\phi}{1 - \sin\phi} + \frac{\cos\phi}{1 + \sin\phi} \right|.$$

and the centre is:

$$\left(0, \frac{1}{2} \left(\frac{-\cos\phi}{1 - \sin\phi} + \frac{\cos\phi}{1 + \sin\phi} \right) \right)$$

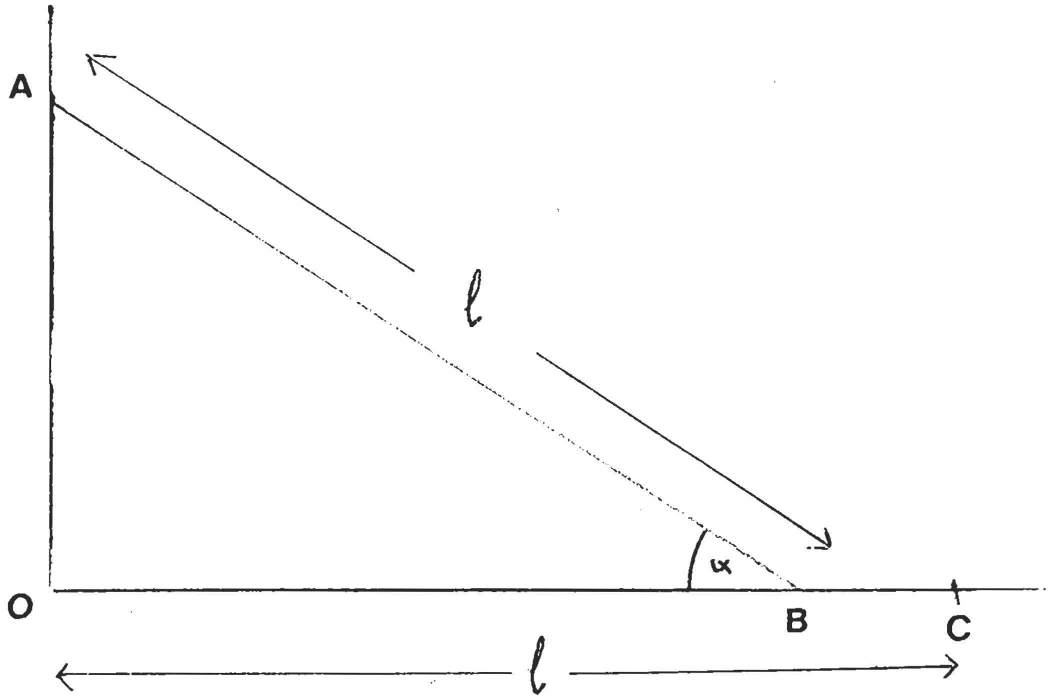
[Hint: Use the results from (a) to determine which points on the line $y = x \tan\phi$ are taken to the y-axis.]

(c) You may assume that C_ϕ goes through (1,0). Using the results from (b), or otherwise, show that the tangent to C_ϕ at (1,0) makes an angle of $\frac{\pi}{2} - \phi$ with the x-axis.

Turn Over

SECTION B

B5.



A plank of length l has one end A fixed on a vertical wall and the other end B resting on horizontal ground in such a way that the side of the plank and the normal to the surface of the plank lie in the same vertical plane. The plank is inclined at an acute angle α to the horizontal. A small smooth stone is released from rest at A and slides down the surface of the plank. Show that its velocity upon reaching the point B is $(2gl\sin\alpha)^{\frac{1}{2}}$, where g is the acceleration due to gravity. Find the time taken for the stone to travel from A to B.

The stone continues to slide along the horizontal ground from B with its horizontal velocity unchanged until it reaches a point C which is at a distance l from the foot O of the wall. Show that the time T taken for the stone to travel from A to C is given by

$$T = \left(\frac{l}{2g\sin\alpha}\right)^{\frac{1}{2}}\left(1 + \frac{1}{\cos\alpha}\right).$$

Treating T as a function of α, show that T has an extreme value when $\cos\alpha = \sqrt{3} - 1$.

B6. A football is kicked with velocity V from a point O on a smooth horizontal pitch in a direction which makes an angle α with the horizontal. You may treat the ball as a point mass and assume that the air resistance is negligible.

(i) Show that the time t_1 taken for the ball to reach the ground is $(2V/g)\sin\alpha$, where g is the acceleration due to gravity. Find the horizontal and vertical velocities of the ball when it reaches the ground.

(ii) Let $e(< 1)$ be the coefficient of restitution between the ball and the ground. Show that the total time t_2 which has elapsed from the kick up to the second bounce is given by

$$t_2 = \frac{2V \sin\alpha}{g}(1+e).$$

(iii) Prove that the total time t_n up to the n-th bounce is given by

$$t_n = \frac{2V \sin\alpha}{g}\left(\frac{1-e^n}{1-e}\right).$$

After what length of time does the ball stop bouncing?

B7. A light inelastic string of length l has one end fixed at a point O and carries a particle of mass m at its other end P. Initially the string is taut and horizontal and the particle is moving vertically downwards with velocity V. Show that the velocity v of the particle when the string makes an acute angle θ with the downward vertical satisfies the equation

$$v^2 = V^2 + 2gl\cos\theta,$$

where g is the acceleration due to gravity.

Show that at some point the string becomes slack if and only if $V^2 < 3gl$.

Now suppose that it is given that $V^2 < 3gl$. Find the speed of the particle at the moment when the string becomes slack. What is the speed of the particle when next it crosses the horizontal line through O ?

B8. A rough circular cylinder of radius a is rigidly attached to a fixed horizontal plane in such a way that the axis of the cylinder is horizontal. A thin uniform plank AB of mass M and length $2l$ carries a load of mass m at A and rests in equilibrium on the cylinder orthogonal to its axis and inclined at an angle α to the horizontal with A higher than B. The distance from A to the point of contact of the plank with the cylinder is d $(< l)$ and neither end of the plank is in contact with the horizontal plane. Draw a diagram of the system showing all the forces which are acting on the plank AB.

Show that:

(i) $d = Ml/(M+m)$;

(ii) the coefficient of friction μ at the point of contact is not less than $\tan\alpha$;

(iii)

$$\frac{(2m+M)l\sin\alpha}{m+M} < a(1+\cos\alpha).$$

SECTION C

C9. A bag contains n_b blue balls, n_r red balls and n_w white balls, where each of these numbers is at least one. Repeatedly a ball is drawn at random from the bag and then replaced.

Let B_t be the number of times in the first t draws that we draw a blue ball, and similarly let R_t be the number of times a red ball is drawn, and let W_t be the number of times a white ball is drawn. What are the distributions of the random variables B_t, R_t and W_t? State their means and variances.

Now let Y_t be the number of times in the first t draws that a blue or red ball is drawn. What is the distribution of Y_t? State its mean and variance. Show that the variance of Y_t is less than the sum of the variances of B_t and R_t.

Let Z be the number of draws until two different colours have been drawn. What is the distribution of $Z - 1$ given that the first ball drawn is blue, and what is the mean of this conditional distribution? Find the mean value of Z.

C10. Let the random variable X be uniformly distributed on the interval $[0, 1]$, and let $Q = \min\{X, 1 - X\}$. Determine the probability $P(Q > t)$ for all values of t; that is, determine the probability $P(X > t \text{ and } 1 - X > t)$.

Now let the random variable Y also be uniformly distributed on $[0, 1]$ and be independent of X; and let $R = \min\{Y, 1-Y\}$. Finally let $S = \min\{Q, R\}$. Find the probability density function, mean and variance of S.

An arrow is shot at a square target which has sides of length 1. The score gained if the arrow hits at point A is four times the smallest of the areas of the four triangles formed by A and the four sides of the target. Show that if we represent the target by the unit square $\{(p, q) : 0 \le p \le 1, 0 \le q \le 1\}$, and if A is the point (x, y), then the score equals $2\min\{x, 1 - x, y, 1 - y\}$. What is the maximum score obtainable?

Assume now that the point A is uniformly distributed over the target: thus if we take A to be the random point (X, Y) in the unit square, then the co-ordinates X and Y are independent and each is uniformly distributed on the interval $[0, 1]$. What is the mean and variance of the score?

C11. A biased coin is tossed repeatedly. On each toss the probability of getting a head is p and of getting a tail is $q = 1 - p$. Let X_i be H or T according to the result of the ith toss. Let Y be the number of tosses until the first tail is seen. (Thus $Y = 1$ corresponds to $X_1 = T$.)

Let E be the event that the first run of r successive heads occurs before the first run of s successive tails. State briefly why we have

$$P(E|Y = j) = P(E|X_1 = T)$$

for each $j = 1, \ldots, r$; and show that

$$P(E|1 < Y \le r) = P(E|X_1 = T).$$

Hence or otherwise prove that

$$P(E|X_1 = H) = p^{r-1} + (1 - p^{r-1})P(E|X_1 = T).$$

Give a similar result for $P(E|X_1 = T)$, and hence determine $P(E)$.

C12. A machine which produces toy ducks (in large numbers) needs occasional expensive adjustments. It is known that when well adjusted the machine produces a proportion $p_0 = 0.1$ of bad ducks.

Suppose that the machine is well adjusted. If we take a sample of n ducks, what is the distribution of the number N of bad ducks observed? State any assumptions you have made. How would you approximate $P(N \geq k)$ if the sample size n is large, and how would you justify this approximation?

If a sample of 100 ducks produced a total of $N = 14$ bad ones, would this indicate that the machine needs adjustment (arguing at the 5% level)?

A new manager wants to institute regular testing. She wants a test that, for appropriately chosen values of n and k, will sample n ducks and report "adjust" if the number N of bad ducks is at least k. If the machine is well adjusted the test must report "adjust" at most 10% of the time, and it must do so at least 95% of the time if the proportion of bad ducks being produced is at least $p_1 = 0.2$. How large must the sample be?

SECTION D

D13. Let a, b and t be real numbers such that $a < b$ and $0 < t < 1$, and let $c = ta + (1-t)b$.

(i) Show that $a < c < b$.

(ii) Given that the number d is such that the three points

$$(a, e^a), \quad (c, d), \quad (b, e^b)$$

lie in a straight line, calculate the value of d.

(iii) Explain, with the aid of a carefully drawn sketch, why

$$e^{ta+(1-t)b} \leq t\, e^a + (1-t)e^b.$$

(iv) Now let a_1, a_2, a_3 be real numbers and let t_1, t_2, t_3 be *positive* real numbers such that

$$t_1 + t_2 + t_3 = 1.$$

By using the fact that

$$t_1 a_1 + t_2 a_2 + t_3 a_3 = (t_1 + t_2)\left(\frac{t_1 a_1 + t_2 a_2}{t_1 + t_2}\right) + t_3 a_3$$

together with the inequality of part (iii), or otherwise, show that

$$e^{t_1 a_1 + t_2 a_2 + t_3 a_3} \leq t_1 e^{a_1} + t_2 e^{a_2} + t^3 e^{a_3}.$$

D14. A sequence

$$N_1, N_2, \ldots, N_{13}$$

of thirteen integers will be said to have property P if every selection of twelve terms from the sequence contains six terms whose combined sum is exactly half the combined sum of the selected twelve.

Suppose that the above sequence does have property P.

(i) Prove that, if one term of the sequence is zero, then all thirteen integers are even.

(ii) Show that if N is any integer then the sequence

$$N_1 - N, N_2 - N, \ldots, N_{13} - N$$

has property P.

(iii) Show also that if k is any integer then the sequence

$$kN_1, kN_2, \ldots, kN_{13}$$

has property P.

(iv) Deduce from these findings, or prove otherwise, that

$$N_1 = N_2 = \ldots = N_{13}.$$

D15. (i) Suppose that n is a positive integer greater than 2 and that 2^k is the highest power of 2 that divides any term of the sequence of positive integers

$$2, 3, \ldots, n.$$

Show that only one term of this sequence is divisible by 2^k.

(ii) Now let b be the lowest common multiple of the numbers

$$2, 3, \ldots, n.$$

Show that $b = 2^k c$, where c is an odd integer.

(iii) By expressing the number

$$x = \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

in the form a/b, or otherwise, show that x cannot be an integer.

D16. (i) A set of weights $w_0, w_1, \ldots, w_n$ of $1, 2, 2^2, \ldots, 2^n$ grams respectively is given. Show that any object weighing less than 2^{n+1} grams, and whose weight is an integral number of grams, can be weighed by putting it in one pan of a balance and a suitable combination of $w_0, w_1, \ldots, w_n$ in the other pan.

(ii) A second set of weights $W_0, W_1, \ldots, W_n$ is given, each of them an integral number of grams. Show that if this new set of weights is used (instead of $w_0, w_1, \ldots, w_n$) to weigh objects in the fashion described in (i) then the number of different object - weights that can be thus weighed is less than or equal to $2^{n+1} - 1$.

(iii) Now suppose further that

(*a*) $1 \leq W_0 \leq W_1 \leq \ldots \leq W_n$,
(*b*) for some $k, W_k \neq 2^k$ grams.

By considering the smallest value of k for which (b) is true and distinguishing carefully the cases $W_k > 2^k$ and $W_k < 2^k$, or otherwise, show that the second set of weights cannot be used to weigh all objects of integral weight less than 2^{n+1} grams.